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Slender streams

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Flows of incompressible inviscid heavy fluids with free or rigid boundary surfaces are considered. For slender streams of fluid, the flow and the free boundaries are represented by a number of different asymptotic expansions in powers of the slenderness ratio. There are three kinds of outer expansions representing respectively jets, which have two free boundaries, wall flows, which have one free and one rigid boundary, and channel flows, which have two rigid boundaries. The flow at the junction of two or more outer flows is represented by an inner expansion. Previously we constructed the three outer expansions and the inner expansion at the junction of a wall flow and a jet (Keller & Geer 1973). Now we construct the inner expansions at the junctions of a channel flow and a jet, a channel flow and a wall flow, and a jet and the two wall flows into which it splits upon hitting a wall. We also match each inner expansion to the adjacent outer expansions. These seven expansions can be combined to solve many problems involving flows of slender streams.

1. Introduction

Previously we presented a method for analysing slender streams of fluid in steady two-dimensional potential flow with gravity acting. It consists in first dividing the stream into one or more long portions connected together by short junctions. Each long part can have either two free boundaries, one free and one rigid boundary, or two rigid boundaries, see figure 1; these parts are called jets, wall flows and channel flows, respectively. There are junction flows which connect a wall flow and a jet (figure 2*a*), a channel flow and a jet (figure 2*b*), a channel flow and a wall flow (figure 2*c*), a jet and the two wall flows into which it splits upon hitting a wall (figure 2*d*), etc. The method proceeds by finding for each part an asymptotic expansion in powers of the slenderness ratio of the stream, and then matching together the expansions in adjoining parts.

The three expansions for the long parts of a stream, called outer expansions, were introduced by Keller & Weitz (1952, 1957). They found the leading coefficients in these expansions. The remaining coefficients were found by Keller & Geer (1973), to be referred to as part 1. In part 1 the expansions of junction flows, called inner expansions, were introduced. Also one of them, the inner expansion at the junction of a wall flow and a jet (figure 2*a*) was found and matched to the adjoining expansions. We shall determine the inner expansions of the other three flows shown in figure 2 and match them to the adjoining expansions.

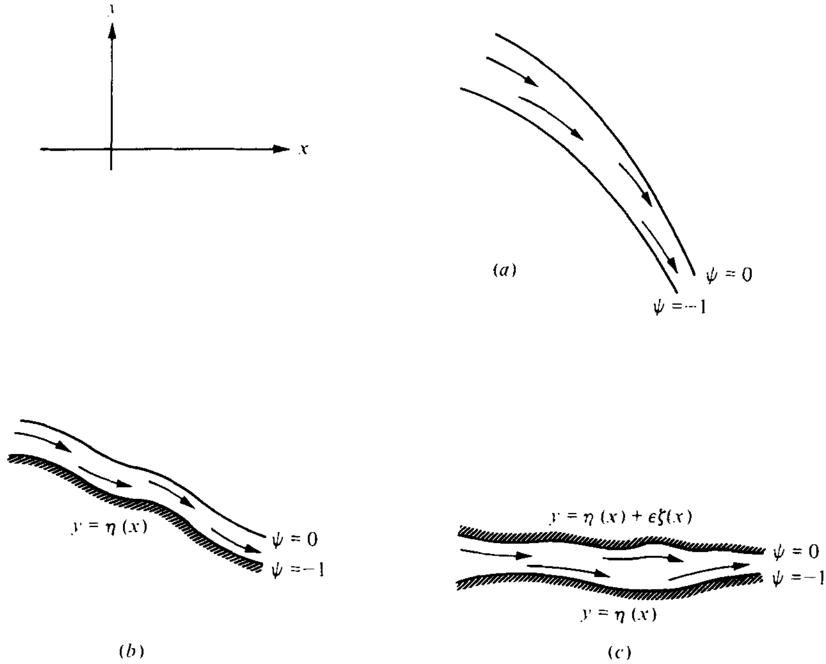

FIGURE 1. The three types of long parts of a stream, which we call outer flows. (*a*) A jet which has two free boundaries. (*b*) A wall flow which has one free boundary and one rigid boundary, indicated by a hatched line, given by $y = \eta(x)$. (*c*) A channel flow which has two rigid boundaries $y = \eta(x)$ and $y = \eta(x) + \epsilon\zeta(x)$. In all cases the upper streamline is $\psi = 0$ and the lower one is $\psi = -1$.

Once these inner expansions have been found, they can be used together with the outer expansions to build up rather complicated flows. However, there are cases in which additional junction flows are needed. Examples of them are the flow in a channel with a corner or with a bifurcation, the flow over a bottom with a step or with a discontinuous change in slope, the impact of a jet on the corner of a wall, etc. Since the leading term in each junction flow is affected neither by gravity nor by wall curvature, it may be given by one of the known special flows which have been found by conformal mapping.

The first application of matched asymptotic expansions to problems involving free streamlines was probably that of Clarke (1965). He treated a waterfall such as that in figure 2(*a*), with a horizontal rigid surface. In that case the slenderness ratio of the stream is just the inverse of the Froude number, so he expanded for large Froude number. Subsequent investigations of slender streams have been made by Bentwich (1968), Clarke (1968), Ackerberg (1968, 1971), Tuck (1976) and Geer (1977*a*, *b*).

Now, before proceeding with our analysis, we shall explain the simple physical idea underlying the present theory. It is that in each cross-section of a slender stream the velocity is nearly constant and parallel to the stream. In a jet or a wall flow this velocity is just that of a freely falling or sliding particle, because the pressure in the flow is nearly constant. Then conservation of mass yields the thickness of the stream. In a channel flow the thickness is given, so conservation of mass yields the velocity.

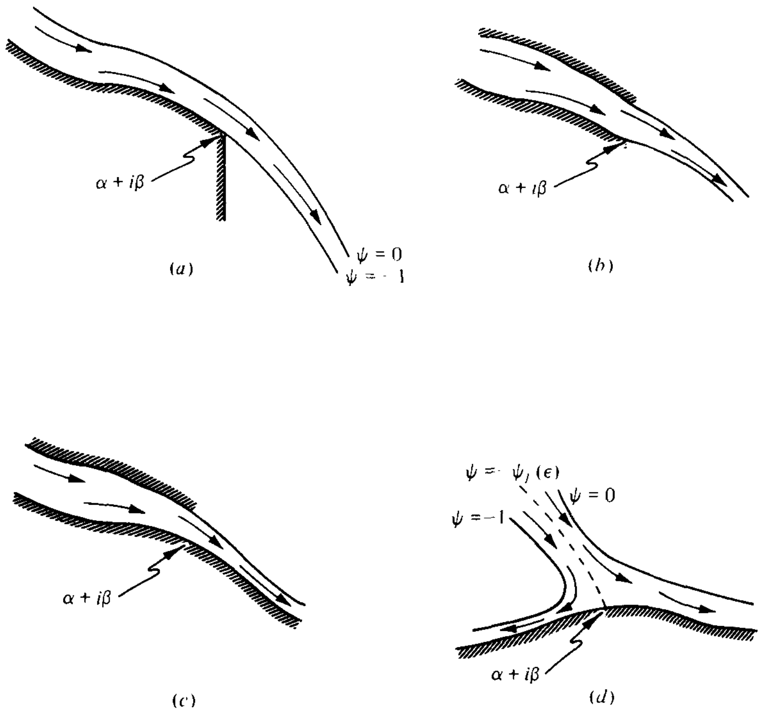

FIGURE 2. Four types of junctions in the physical plane: (*a*) a wall flow becoming a jet; (*b*) a channel flow becoming a jet; (*c*) a channel flow becoming a wall flow; (*d*) a jet splitting into two wall flows upon hitting a wall. The point (α, β) and the streamlines $\psi = 0$ and $\psi = -1$ are shown, as is the dividing streamline $\psi = -\psi^0$ in case (*d*).

Thus in the first two cases the flow can be found approximately by solving the ordinary differential equations for a falling or sliding particle, while the third case is even simpler. As was shown in part 1, these ordinary differential equations are the only nonlinear equations which arise in finding the outer expansions.

In § 2 we formulate the method. Then in § 3 we determine the leading coefficient in the inner expansions of flows *b* and *c* shown in figure 2. In §§ 4 and 5 we determine the subsequent coefficients in these two expansions. In § 6 we match them to the appropriate outer expansions. In § 7 we treat case *d* of figure 2. Finally, in § 8 we give a summary of our results. We also present figures showing a complicated flow which we have calculated by using our method. The flow emerges from a channel, flows along a wall, leaves the wall and becomes a falling jet, hits the ground and splits into two wall flows. See figure 4. The details of this calculation, as well as other applications of our method, will be presented elsewhere.

2. Formulation of the method

Let us consider a steady potential flow in the x', y' plane bounded by the streamlines $\psi' = 0$ and $\psi' = -hU$. Here h is a typical width of the stream and U is a typical value of its velocity. In this flow $z' = x' + iy'$ is an analytic function of the complex potential $\phi' + i\psi'$. This analytic function is defined in the strip $-hU \leqslant \psi' \leqslant 0$, and must satisfy suitable conditions at $\phi' = \pm\infty$. In addition on each part of a fixed streamline, i.e. a bounding streamline corresponding to a rigid boundary, z' must lie on that boundary. On a free portion of a streamline z' must satisfy the constant pressure condition $|dz'/d\phi'|^{-2}+2gy' = U^2$. Here g is the acceleration of gravity.

In order to determine $z'(\phi'+i\psi')$ we first introduce dimensionless variables. Let L be a typical length along the stream, and let us define dimensionless quantities by

$$z' = Lz, \quad \phi' = LU\phi, \quad \psi' = hU\psi, \quad \gamma = 2gL/U^2, \quad \epsilon = h/L. \tag{2.1}$$

Here γ^{-1} is the Froude number of the flow and ϵ is the slenderness ratio of the stream.† In these variables the flow is determined by the analytic function $z(\phi+i\epsilon\psi,\epsilon)$, which depends explicitly upon ϵ and is defined in the strip $-1 \leqslant \psi \leqslant 0$. The corresponding boundary conditions are

$$|dz/d\phi|^2 = (1-\gamma\,\mathrm{Im}\,z)^{-1} \quad \text{on a free streamline,} \tag{2.2}$$

$$\mathrm{Im}\,z = \eta(\mathrm{Re}\,z) \quad \text{on the fixed streamline} \quad y = \eta(x), \tag{2.3}$$

$$\mathrm{Im}\,z = \eta(\mathrm{Re}\,z)+\epsilon\zeta(\mathrm{Re}\,z) \quad \text{on the fixed streamline} \quad y = \eta(x)+\epsilon\zeta(x). \tag{2.4}$$

We suppose that the strip can be divided into intervals $\phi_j(\epsilon) < \phi < \phi_{j+1}(\epsilon)$, such that throughout each interval just one of these conditions applies on each bounding streamline. An interval represents a jet if (2.2) holds on both streamlines, a wall flow if (2.2) holds on one streamline and (2.3) or (2.4) holds on the other, and a channel flow if (2.3) holds on one and (2.4) on the other. Within each interval we assume that z has an (outer) asymptotic expansion in powers of ϵ of the form

$$z(\phi+i\epsilon\psi,\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n z_n(\phi+i\epsilon\psi), \quad \phi_j(\epsilon) < \phi < \phi_{j+1}(\epsilon). \tag{2.5}$$

The fact that z_n depends upon j will be indicated only when it is necessary to do so. The outer expansion coefficients $z_n(\phi) = x_n(\phi)+iy_n(\phi)$ were found in part 1 for all three cases.

The outer expansions do not hold at the endpoints $\phi_j(\epsilon)$ of the intervals in which they are defined. Therefore for ϕ in the neighbourhood of $\phi_j(\epsilon)$ we assume that z can be represented by an inner asymptotic expansion. To write it we choose a point $z = \alpha+i\beta$ at or near the junction in the z plane and denote by $\phi_j(\epsilon)+i\epsilon\psi_j(\epsilon)$ the complex potential there:

$$z[\phi_j(\epsilon)+i\epsilon\psi_j(\epsilon),\epsilon] = \alpha+i\beta. \tag{2.6}$$

Then we introduce the stretched variables ϕ'' and z'' defined by

$$\phi'' = [\phi-\phi_j(\epsilon)]/\epsilon, \quad z''(\phi''+i\psi,\epsilon) = (z-\alpha-i\beta)/\epsilon. \tag{2.7}$$

From (2.6) and (2.7) we see that $\phi'' = 0$ at $\phi = \phi_j(\epsilon)$ and that $z'' = 0$ at $\phi'' = 0$, $\psi = \psi_j(\epsilon)$. Now we write the inner expansion as

$$z''(\phi''+i\psi,\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n z''_n(\phi''+i\psi). \tag{2.8}$$

† For a jet, or for flow along a straight wall or in a straight channel, the only choice for L is a multiple of $U^2/2g$. With $L = U^2/2g$ we have $\gamma = 1$ and $\epsilon = 2gh/U^2$, so that ϵ^{-1} is the Froude number. Then the stream is slender only at high Froude numbers, which is the case treated by Clarke (1968). However, for flows along curved walls or in curved channels, a flow can be slender at any Froude number.

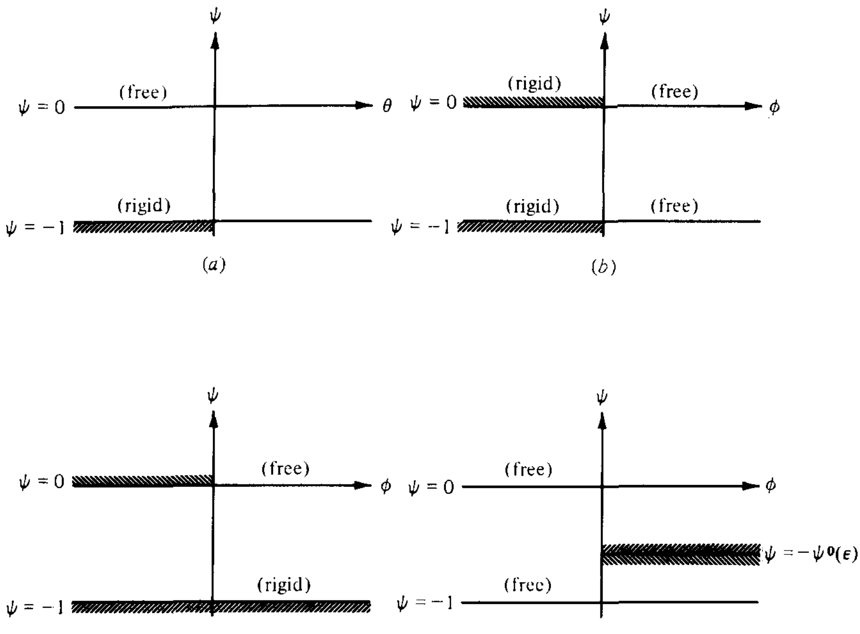

FIGURE 3. Four junction flows in the plane of the complex potential $\phi+i\psi$, corresponding to the flows shown in the physical plane in figure 2. Each flow is bounded by the streamlines $\psi = 0$ and $\psi = -1$. The point α, β in the physical plane corresponds to $(\phi_j(\epsilon), -1)$ in cases (*a*), (*b*) and (*c*), and to $(\phi_j(\epsilon), -\psi^0(\epsilon))$ in case (*d*). The portion of each streamline which lies on a rigid boundary in the physical plane is shown hatched, while the remaining portions denote free streamlines.

The coefficients z''_n in (2.8) are analytic functions of $\phi''+i\psi$ defined in the strip $-1 \leqslant \psi \leqslant 0$. The boundary conditions which they satisfy can be obtained from (2.2)–(2.4) by using (2.7). The conditions and the kind of streamline on which each holds are

$$|dz''/d\phi''|^2 = (1-\gamma\beta-\epsilon\gamma y'')^{-1}, \quad \text{free}, \tag{2.9}$$

$$\beta+\epsilon y'' = \eta(\alpha+\epsilon x''), \quad \text{fixed} \quad y = \eta(x), \tag{2.10}$$

$$\beta+\epsilon y'' = \eta(\alpha+\epsilon x'')+\epsilon\zeta(\alpha+\epsilon x''), \quad \text{fixed} \quad y = \eta(x)+\epsilon\zeta(x). \tag{2.11}$$

We shall now specify the boundary conditions and the values of α, β, $\phi_j(\epsilon)$ and $\psi_j(\epsilon)$ for each of the last three cases shown in figure 2, referring to them as cases *b*, *c* and *d*. The first case, *a*, was treated in part 1, §§ 6 and 7. By referring to figure 2 and the definition (2.7) of the new variables, we obtain the following (see figure 3).

Case b: (2.9) holds on the two free streamlines $\psi = -1$, $\phi'' > 0$ and $\psi = 0$, $\phi'' > 0$;
(2.10) holds on the fixed streamline $\psi = -1$, $\phi'' < 0$;
(2.11) holds on the fixed streamline $\psi = 0$, $\phi'' < 0$;

$$\alpha = \beta = \phi_j(\epsilon) = 0, \quad \psi_j(\epsilon) = -1.$$

Case c: (2.9) holds on the free streamline $\psi = 0, \phi'' > 0$;
(2.10) holds on the fixed streamline $\psi = -1$;
(2.11) holds on the fixed streamline $\psi = 0$, $\phi'' < 0$;

$$\alpha = \beta = \phi_j(\epsilon) = 0, \quad \psi_j(\epsilon) = -1.$$

Case d: (2.9) holds on the free streamlines $\psi = 0$ and $\psi = -1$.

(2.10) holds on both sides of the fixed streamline $\psi = \psi_j(\epsilon), \phi > \phi_j(\epsilon) = 0$, which slits the strip $-1 < \psi < 0$. See figure 3(*d*).

$$\phi_j(\epsilon) = 0; \quad \psi_j(\epsilon) = -\psi^0(\epsilon), \quad \alpha \text{ and } \beta \text{ to be determined.}$$

In all cases $\beta = \eta(\alpha)$.

From (2.9) we see that the dimensionless speed of the flow, $|d\phi''/dz''|$, is equal to unity on a free streamline at $y'' = -\epsilon^{-1}\beta$. This shows that the typical velocity U is just the dimensional velocity at this place. In cases a and b, the free part of the streamline $\psi = -1$ starts at $\alpha = \beta = 0$, so U is the flow velocity there. Then, since the flux in the stream is hU, the stream width there is practically h, and tends to h as h tends to zero. In cases c and d, U is the flow velocity on a free streamline at the height of the point $z = \alpha + i\beta$, which corresponds to $z'' = 0$.

3. Expansion of boundary conditions and determination of z_0

As the first step in determining the coefficients in (2.8), we substitute (2.8) into the boundary conditions (2.9)–(2.11). In doing so, and from now on, we shall omit the double primes and use a prime to denote differentiation. Then we equate coefficients of each power of ϵ in the resulting equations. This calculation is carried out in part 1, § 6, for the two conditions (2.9) and (2.10) with $\alpha = \beta = 0$. The results from (2.9) are given by (6.5) and (6.6) of part 1, which are

$$x_0'^2 + y_0'^2 = 1, \tag{3.1}$$

$$x_0' x_k' + y_0' y_k' = J_k, \quad k = 1, 2, \ldots . \tag{3.2}$$

Here J_k, given in (6.6) of part 1, involves x_n and y_n with $n < k$. From (2.10) we obtain (6.8) of part 1, which is

$$y_k - \eta'(0)\, x_k = K_k, \quad k = 0, 1, \ldots . \tag{3.3}$$

Again K_k, given in (6.8) of part 1, involves x_n and y_n with $n < k$, and $K_0 = 0$.

By using (2.8) in (2.11) and equating coefficients of powers of ϵ we obtain

$$y_k - \eta'(0)\, x_k = L_k, \quad k = 0, 1, \ldots . \tag{3.4}$$

Here L_k is given by

$$L_k = K_k + \sum_{j=1}^{k} \frac{1}{j!} \zeta^{(j)}(0) \sum_{n_1 + \ldots + n_j = k-j} x_{n_1} \cdots x_{n_j}. \tag{3.5}$$

We must now find analytic functions $z_n(\phi + i\psi)$ defined in the strip $-1 \leqslant \psi \leqslant 0$ satisfying (3.1)–(3.4) on the appropriate boundaries.†

In case b, (3.1) and (3.2) must hold on $\psi = -1$, $\phi > 0$ and $\psi = 0$, $\phi > 0$; (3.3) on $\psi = -1$, $\phi < 0$ and (3.4) on $\psi = 0$, $\phi < 0$. In case c, (3.1) and (3.2) hold on $\psi = 0$, $\phi > 0$; (3.3) on $\psi = -1$ and (3.4) on $\psi = 0$, $\phi < 0$. In case d, (3.1) and (3.2) hold on $\psi = 0$ and on $\psi = -1$ while (3.3) holds on both sides of $\psi = \psi_j(\epsilon)$, $\phi > \phi_j(\epsilon)$.

† For arbitrary α and β, (3.1)–(3.5) are changed as follows. In (3.1) the right-hand side is multiplied by $(1-\gamma\beta)^{-1}$. In J_k on the right-hand side of (3.2) the factor γ^j in (6.6) of part 1 is multiplied by $(1-\gamma\beta)^{-j-1}$. In (3.3) and (3.4) η' and the $\eta^{(j)}$ in K_k and L_k are evaluated at α. In (3.5) $\zeta^{(j)}$ is evaluated at α.

We shall first determine $z_0 = x_0 + iy_0$ in case b. From the preceding paragraph we find that z_0 must satisfy the following boundary conditions:

$$x_0'^2 + y_0'^2 = 1 \quad \text{on} \quad \psi = 0, \quad \phi > 0 \quad \text{and on} \quad \psi = -1, \quad \phi > 0; \tag{3.6}$$

$$y_0 = \eta'(0)\, x_0 \quad \text{on} \quad \psi = -1, \quad \phi < 0; \tag{3.7}$$

$$y_0 = \eta'(0)\, x_0 + \zeta(0) \quad \text{on} \quad \psi = 0, \quad \phi < 0. \tag{3.8}$$

In addition z_0 must be analytic in the strip $0 > \psi > -1$ with $z_0 = 0$ at $\phi = 0, \psi = -1$. A solution of this problem is

$$z_0 = e^{i\theta}(\phi + i\psi + i), \quad \text{where} \quad \tan\theta = \eta'(0). \tag{3.9}$$

In component form

$$x_0 = \phi\cos\theta - (\psi + 1)\sin\theta, \quad y_0 = \phi\sin\theta + (\psi + 1)\cos\theta. \tag{3.10}$$

It is evident that z_0 given by (3.9) is analytic in the strip, vanishes at $\phi = 0, \psi = -1$, and satisfies (3.6) and (3.7). To show that it satisfies (3.8) we set $\psi = 0$ in (3.10) and write

$$\begin{aligned} y_0 - \eta'(0)\, x_0 &= \phi\sin\theta + \cos\theta - \tan\theta(\phi\cos\theta - \sin\theta) \\ &= \cos\theta + \tan\theta\sin\theta = \sec\theta = \zeta(0). \end{aligned} \tag{3.11}$$

The last equality in (3.11) follows from the fact that the dimensionless flux in the stream is unity and the dimensionless velocity $|d\phi/dz_0|$ is also unity, so the normal width of the stream must be unity. This width is just $\zeta(0)\cos\theta$ so $\zeta(0) = \sec\theta$, which is the value we used above. Thus (3.8) is satisfied.

A completely similar analysis shows that (3.9) or (3.10) is the solution for z_0 in case c. It is also the solution in case a, as (6.10) of part 1 shows. This solution represents a stream of constant width in which the flow velocity is uniform. The solution z_0 for case d is more complicated and will be given later.

4. Determination of z_k in case b (channel to jet)

The coefficient z_k, $k \geqslant 1$, is an analytic function of $\phi + i\psi$ in the strip $-1 \leqslant \psi \leqslant 0$. On the boundaries of the strip it satisfies the inhomogeneous boundary conditions (3.2)–(3.4), as is explained in the paragraph following (3.5). To find z_k in case b we find it convenient to introduce w_k defined by

$$w_k = e^{-i\theta} z_k, \tag{4.1}$$

where $\tan\theta = \eta'(0)$. In terms of w_k the boundary conditions for case b are for $k \geqslant 1$:

$$\operatorname{Re} w_k' = J_k, \quad \text{on} \quad \psi = 0, \quad \phi > 0 \quad \text{and} \quad \psi = -1, \quad \phi > 0. \tag{4.2}$$

$$\operatorname{Im} w_k' = K_k' \cos\theta, \quad \text{on} \quad \psi = -1, \quad \phi < 0, \tag{4.3}$$

$$\operatorname{Im} w_k' = L_k' \cos\theta, \quad \text{on} \quad \psi = 0, \quad \phi < 0. \tag{4.4}$$

In (4.3) and (4.4) we have differentiated the boundary condition with respect to ϕ in order to obtain conditions involving only w_k'.

We must now find a function w_k' analytic in the strip $-1 \leqslant \psi \leqslant 0$ and satisfying

(4.2)–(4.4). To do so, we map the strip onto the first quadrant of the plane of a new complex variable s defined by

$$s = \left(\frac{1+e^{-\pi f}}{1-e^{-\pi f}}\right)^{\frac{1}{2}}, \quad f = \phi + i\psi. \tag{4.5}$$

Then we solve for w_k' as a function of s. A particular solution is

$$w_k'(\phi+i\psi) = \frac{2is}{\pi}\int_0^\infty \left[\frac{p_k(\sigma)}{s^2+\sigma^2}+\frac{q_k(\sigma)}{s^2-\sigma^2}\right] d\sigma. \tag{4.6}$$

Here p_k and q_k are defined by

$$\begin{aligned}
p_k(\sigma) &= K_k'(\phi,-1)\cos\theta = K_k'(\pi^{-1}\log[(1-\sigma^2)/(1+\sigma^2)],-1)\cos\theta, \quad 0<\sigma<1,\\
p_k(\sigma) &= L_k'(\phi,0)\cos\theta = L_k'(\pi^{-1}\log[(\sigma^2-1)/(1+\sigma^2)],0)\cos\theta, \quad 1<\sigma<\infty,\\
q_k(\sigma) &= J_k(\phi,-1) = J_k(-\pi^{-1}\log[(1-\sigma^2)/(1+\sigma^2)],-1), \quad 0<\sigma<1,\\
q_k(\sigma) &= J_k(\phi,0) = J_k(-\pi^{-1}\log[(\sigma^2-1)/(1+\sigma^2)],0), \quad 1<\sigma<\infty.
\end{aligned} \tag{4.7}$$

The solution w_k' is not unique. We can add to it any linear combination of the eigenfunctions described in appendix A. However, those functions are all singular either at $z=0$ or at $z=\pm\infty$. Therefore, in matching, their coefficients would be found to vanish, so they have been omitted.

We now use (4.1) to obtain $z_k' = e^{i\theta}w_k'$, with w_k' given by (4.6). Then we integrate with respect to $f=\phi+i\psi$, recalling that $z_k=0$ at $\phi=0$, $\psi=-1$. In this way we obtain

$$z_k(\phi+i\psi) = \frac{2i\,e^{i\theta}}{\pi}\int_{-i}^{\phi+i\psi} s\int_0^\infty \left[\frac{p_k(\sigma)}{s^2+\sigma^2}+\frac{q_k(\sigma)}{s^2-\sigma^2}\right] d\sigma\, df. \tag{4.8}$$

Here s is given by (4.5).

For $k=1$, we find from (6.6) of part 1 that $J_1 = \gamma y_0/2$, from (6.8) of part 1 that $K_1 = \eta''(0)x_0^2/2$ and from (3.5) that $L_1 = \eta''(0)x_0^2/2+\zeta'(0)x_0$. Thus $K_1' = \eta''(0)x_0\cos\theta$ and $L_1' = \eta''(0)x_0\cos\theta+\zeta'(0)\cos\theta$. We use these expressions in (4.7) and use the results in (4.8). The σ integration in (4.8) can then be performed and after some calculation we obtain the following result for z_1, in which s is defined by (4.5):

$$\begin{aligned}
z_1(\phi+i\psi) = \frac{e^{i\theta}}{2\pi}\int_{-i}^{\phi+i\psi} \Bigg\{&\gamma\sin\theta\log\left[\frac{(1-is)^2}{1-s^2}\right]+\gamma\cos\theta\left[\pi+i\log\left(\frac{s-1}{s+1}\right)\right]\\
&-2i\eta''(0)\cos^3\theta\log\left[\frac{1+2s+s^2}{1+s^2}\right]\\
&-2[\eta''(0)\cos^2\theta\sin\theta-\cos^2\theta\zeta'(0)]\log\left(\frac{1+is}{1-is}\right)\Bigg\} df.
\end{aligned} \tag{4.9}$$

The asymptotic behaviour of z as ϕ tends to $\pm\infty$, obtained from (4.9), is given by

$$\begin{aligned}
z_1(\phi+i\psi) \sim -\frac{i\gamma}{4}e^{2i\theta}f^2+\frac{e^{i\theta}}{4\pi}\{&\gamma[\pi e^{i\theta}+\cos\theta(\pi-i\log 4)]\\
&-2i\eta''(0)\cos^2\theta(\cos\theta\log 4+\pi\sin\theta)+2\pi i\cos^2\theta\zeta'(0)\}f+C_{10}^{+}+O(e^{-\pi f}),\\
&\text{as}\quad \phi = \mathrm{Re}\,f\to\infty,
\end{aligned} \tag{4.10}$$

$$z_1(\phi+i\psi) \sim \frac{e^{i\theta}\cos^2\theta}{2}\{\zeta'(0)-i\eta''(0)\,e^{i\theta}\}(\phi+i\psi)^2$$
$$+\frac{e^{i\theta}}{4\pi}\{\gamma\,(\log 4\sin\theta+\pi\cos\theta)+2\eta''(0)\cos^2\theta(2\pi e^{i\theta}+\pi\cos\theta+\log 4\sin\theta)$$
$$-4\pi(\pi i+\log 2)\cos^2\theta\zeta'(0)\}(\phi+i\psi)$$
$$+C_{10}^{-}+O(e^{\pi f}), \quad \text{as} \quad \operatorname{Re} f=\phi\to-\infty. \tag{4.11}$$

Here $C_{10}^{\pm}$ are constants whose values can be obtained from (4.9), but which we will not need.

5. The z_k in case c (channel to wall flow)

In case c the boundary conditions on w_k' are found to be

$$\operatorname{Re} w_k' = J_k \quad \text{on} \quad \psi=0, \quad \phi>0, \tag{5.1}$$
$$\operatorname{Im} w_k' = K_k'\cos\theta \quad \text{on} \quad \psi=-1, \tag{5.2}$$
$$\operatorname{Im} w_k' = L_k'\cos\theta \quad \text{on} \quad \psi=0, \quad \phi<0. \tag{5.3}$$

To find w_k' we proceed as in § 4, but now we map the strip onto the second quadrant of the s plane by the mapping

$$s=(e^{\pi f}-1)^{\frac{1}{2}}, \quad f=\phi+i\psi. \tag{5.4}$$

Again we obtain (4.6) for w_k' and (4.8) for z_k, but with s given by (5.4) and the following values of $p_k(\sigma)$ and $q_k(\sigma)$:

$$p_k(\sigma)=L_k'(\phi,0)\cos\theta=L_k'(\pi^{-1}\log[1-\sigma^2],0)\cos\theta, \qquad 0<\sigma<1;$$
$$p_k(\sigma)=K_k'(\phi,-1)\cos\theta=K_k'(\pi^{-1}\log[\sigma^2-1],-1)\cos\theta, \quad 1<\sigma<\infty;$$
$$q_k(\sigma)=J_k(\phi,0)=J_k(\pi^{-1}\log[\sigma^2+1],0), \qquad 0\leqslant\sigma<\infty. \tag{5.5}$$

For $k=1$ we find

$$z_1(\phi+i\psi)=\frac{ie^{i\theta}}{2}\cos^3\theta\eta''(0)\{(\phi+i\psi)^2+1\}+e^{i\theta}\cos\theta\left[\frac{\gamma}{2}+i\cos\theta(\zeta'(0)-\eta''(0)\sin\theta)\right]$$
$$\times(\phi+i\psi+i)-\frac{\cos^2\theta}{\pi}(\zeta'(0)-\eta''(0)\sin\theta)\,e^{i\theta}\int_{-i}^{\phi+i\psi}\log[1+i(e^{\pi f}-1)^{\frac{1}{2}}]\,df$$
$$+\frac{e^{i\theta}}{\pi}[\gamma\sin\theta+\cos^2\theta(\zeta'(0)-\eta''(0)\sin\theta)]\int_{-i}^{\phi+i\psi}\log[1-i(e^{\pi f}-1)^{\frac{1}{2}}]\,df. \tag{5.6}$$

From (5.6) we obtain the asymptotic results

$$z_1(\phi+i\psi)\sim\tfrac{1}{2}e^{i\theta}[\gamma\sin\theta+i\cos^3\theta\eta''(0)](\phi+i\psi)^2$$
$$+[\tfrac{1}{2}\gamma-2i\,e^{i\theta}(\zeta'(0)-\eta''(0)\sin\theta)\cos^2\theta](\phi+i\psi)$$
$$+C_{10}^{+}+O(e^{-\frac{1}{2}\pi f}), \quad \text{as} \quad \phi=\operatorname{Re} f\to\infty, \tag{5.7}$$

$$z_1(\phi+i\psi)\sim\frac{\cos^2\theta}{2}(i\eta''(0)-e^{i\theta}\zeta'(0))(\phi+i\psi)^2+\frac{e^{i\theta}}{2\pi}\{\gamma(\pi\cos\theta+2\log 2\sin\theta)$$
$$+(\zeta'(0)-\eta''(0)\sin\theta)(\cos^2\theta\log 16)\}(\phi+i\psi)+C_{10}^{-}+O(e^{\pi f}),$$
$$\text{as} \quad \phi=\operatorname{Re} f\to-\infty. \tag{5.8}$$

The non-uniqueness of w_k' is considered and clarified in appendix A. The constants $C_{10}^{\pm}$ can be obtained from (5.6), but we shall not need them.

6. Matching

To complete the analysis of the flows in figure 2 we must match each inner expansion to the appropriate outer expansions. For case a this was done in part 1, § 7. We shall now do it for the other cases, using the following form of the matching principle:

$$I_N(O_N) = O_N(I_N). \tag{6.1}$$

Here $I_N(O_N)$ is the $N+1$ term inner expansion of the $N+1$ term outer expansion, and $O_N(I_N)$ is the $N+1$ term outer expansion of the $N+1$ term inner expansion.

In order to compute $I_N(O_N)$ we write the $N+1$ term outer expansion O_N of

$$z(\phi + i\epsilon\psi, \epsilon).$$

Next we write the outer variable $\phi + i\epsilon\psi$ in terms of the inner variable $f = \phi'' + i\psi$ thus: $\phi + i\epsilon\psi = \phi_k(\epsilon) + \epsilon f$. Then we expand each term in powers of ϵ and finally retain $N+1$ terms. When $\phi_k(\epsilon) = 0$, the successive steps are

$$\begin{aligned} z(\phi + i\epsilon\psi, \epsilon) \sim \sum_{n=0}^{N} z_n(\phi + i\epsilon\psi)\,\epsilon^n = \sum_{n=0}^{N} z_n(\epsilon f)\,\epsilon^n &= \sum_{n=0}^{N} \left[\sum_{j=0}^{\infty} \frac{1}{j!} z_n^{(j)}(0) f^j \epsilon^j\right] \epsilon^n \\ &= \sum_{q=0}^{N} \left[\sum_{j=0}^{q} \frac{1}{j!} z_{q-j}^{(j)}(0) f^j\right] \epsilon^q + O(\epsilon^{N+1}). \end{aligned} \tag{6.2}$$

Thus

$$I_N(O_N) = \sum_{q=0}^{N} \left[\sum_{j=0}^{q} \frac{1}{j!} z_{q-j}^{(j)}(0) f^j\right] \epsilon^q. \tag{6.3}$$

Now to compute $O_N(I_N)$ we write the $N+1$ term inner expansion of $z[\phi_k(\epsilon) + \epsilon f, \epsilon]$. Then we write f in terms of the outer variable $\tau = \phi + i\epsilon\psi$ in the form $f = \epsilon^{-1}[\tau - \phi_k(\epsilon)]$. Next we expand each term for ϵ small and retain $N+1$ terms. Finally we rewrite the result in terms of the inner variable f. The first steps are as follows, when $\phi_k(\epsilon) = 0$:

$$z(\epsilon f, \epsilon) = z(0,0) + \epsilon z''(f, \epsilon) \sim z(0,0) + \epsilon \sum_{n=0}^{N-1} z_n''(f)\,\epsilon^n = z(0,0) + \epsilon \sum_{n=0}^{N-1} z_n''(\epsilon^{-1}\tau)\,\epsilon^n. \tag{6.4}$$

We recall that $z(0,0) = \alpha + i\beta$.

To proceed further we need the expansion of $z_n''(\epsilon^{-1}\tau)$ as ϵ tends to zero, or equivalently the expansion of $z_n''(f)$ as f tends to infinity. Now within the strip in which z_n'' is defined, $\mathrm{Im} f$ is bounded but $\mathrm{Re} f$ can tend to $\pm\infty$. As we have shown in cases a, b, and c, and will show in the next section for case d, for $n = 0$ and $n = 1$, $z_n''(f)$ has an expansion of the form

$$z_n''(f) = \sum_{j=0}^{n+1} C_{nj}^{\pm} f^j + O(e^{-a|f|}) \quad \text{as} \quad \mathrm{Re} f \to \pm\infty. \tag{6.5}$$

Here the $C_{nj}^{\pm}$ are constants and a is some positive number. This result (6.5) can be proved for all n by induction.

We can now use (6.5) in (6.4) and perform the last steps to obtain

$$\begin{aligned} z(\epsilon f, \epsilon) &\sim z(0,0) + \epsilon \sum_{n=0}^{N-1} \left[\sum_{j=0}^{n+1} C_{nj}^{\pm} \tau^j \epsilon^{-j}\right] \epsilon^n \\ &= z(0,0) + \sum_{q=1}^{N} \left[\sum_{j=0}^{q} C_{q-1,j}^{\pm} f^j\right] \epsilon^q. \end{aligned} \tag{6.6}$$

Thus we have

$$O_N(I_N) = z(0,0) + \sum_{q=1}^{N}\left[\sum_{j=0}^{q} C^{\pm}_{q-1,j} f^j\right]\epsilon^q \quad \text{as} \quad \mathrm{Re} f \to \pm\infty. \tag{6.7}$$

We next substitute (6.3) and (6.7) into (6.1), choosing the superscript plus or minus according as we are matching at $\mathrm{Re} f = +\infty$ or $-\infty$ in the inner expansion. Then we equate coefficients of like powers of ϵ to get

$$z_0(0) = z(0,0), \tag{6.8}$$

$$\sum_{j=0}^{q} \frac{1}{j!} z^{(j)}_{q-j}(0) f^j = \sum_{j=0}^{q} C^{\pm}_{q-1,j} f^j, \quad q \geqslant 1. \tag{6.9}$$

By equating coefficients of like powers of f in (6.9) we obtain

$$z^{(j)}_{n-j}(0) = j!\, C^{\pm}_{n-1,j}, \quad n \geqslant 1, \quad j = 0, \ldots, n. \tag{6.10}$$

Equations (6.8) and (6.10) are the results of matching.

By choosing for n and j the values 0, 1, 2 we obtain from (6.10) the following particular results:

$$z^{(1)}_0(0) = C^{\pm}_{01}; \quad z_1(0) = C^{\pm}_{00}; \qquad (6.11), (6.12)$$

$$z^{(1)}_1(0) = C^{\pm}_{11}; \quad z^{(2)}_0(0) = 2C^{\pm}_{12}. \qquad (6.13), (6.14)$$

The constants $C^{\pm}_{nj}$ are completely determined by each inner expansion. Since we have evaluated z_0'' and z_1'' explicitly and expanded them for $\mathrm{Re} f \to \pm\infty$, we have shown how to obtain $C^{\pm}_{00}$, $C^{\pm}_{01}$, $C^{\pm}_{10}$, $C^{\pm}_{11}$ and $C^{\pm}_{12}$. These are just the constants which occur in (6.11)–(6.14). In case a, these $C^{\pm}_{nj}$ are the coefficients of powers of $\phi + i\psi$ in (6.10), (6.21) and (6.22) of part 1. When we use them in (6.11)–(6.14) of this paper we recover exactly the matching conditions (7.8) and (7.10) obtained in part 1 in a different way.

When the outer expansion is that of a wall flow or a channel flow, z_0 is completely determined by specifying $x_0(0)$, as we see from equations (4.8) and (5.11) of part 1 respectively. The matching condition (6.8) yields this value. The second term z_1 is determined by specifying $x_1(0)$, as we see from (4.15) and (4.16) of part 1 for a wall flow, and from (5.15) and (5.16) of part 1 for a channel flow. The matching condition (6.12) yields $x_1(0) = \mathrm{Re}\, C^{\pm}_{00}$. Thus only $C^{\pm}_{00}$ need be found from the inner expansion to determine z_0 and z_1 in an adjacent wall flow or channel flow, and only z_0'' is needed for this purpose.

If the outer expansion is that of a jet, we need both $z_0(0)$ and $z^{(1)}_0(0)$ to determine z_0, and both $z_1(0)$ and $z^{(1)}_1(0)$ to determine z_1, as we see from part 1, § 3. Now $z_0(0)$ and $z^{(1)}_0$ are given by (6.8) and (6.11) in terms of $z(0,0)$ and $C^{\pm}_{01}$, while $z_1(0)$ and $z^{(1)}_1(0)$ are given by (6.12) and (6.13) in terms of $C^{\pm}_{00}$ and $C^{\pm}_{11}$. Thus the constants $C^{\pm}_{00}$ and $C^{\pm}_{01}$ from z_1'' and $C^{\pm}_{11}$ from z_1'' are needed to get z_0 and z_1 in a jet.

For case a (wall flow to jet) we obtain the following values from the equations of part 1, $C^{\pm}_{0j}$ from (6.10), C^{+}_{1j} from (6.21) and C^{-}_{1j} from (6.22):

$$\left.\begin{aligned}
&C^{\pm}_{00} = i e^{i\theta}, \quad C^{\pm}_{01} = e^{i\theta}, \quad C^{\pm}_{0j} = 0, \; j \geqslant 2,\\
&C^{+}_{11} = \frac{e^{i\theta}}{2\pi}(\pi\gamma e^{i\theta} - i\cos\theta \log 4[2\eta''(0)\cos^2\theta + \gamma]), \quad C^{+}_{12} = -\frac{i\gamma e^{2i\theta}}{4},\\
&C^{-}_{11} = \tfrac{1}{2}\gamma e^{2i\theta}, \quad C^{-}_{12} = \tfrac{1}{2}e^{i\theta}(\tfrac{1}{2}\gamma\sin\theta + i\eta''(0)\cos^3\theta).
\end{aligned}\right\} \tag{6.15}$$

For case b (channel flow to jet) we obtain from (3.9) the same values of $C^{\pm}_{0j}$ as in (6.15). We also obtain the following values of C^{+}_{1j} from (4.10) and of C^{-}_{1j} from (4.11):

$$\left.\begin{aligned} C^{+}_{11} &= (4\pi)^{-1}e^{i\theta}[\gamma(\pi e^{i\theta}+[\pi-i\log 4]\cos\theta) \\ &\qquad -2i\eta''(0)\cos^2\theta(\log 4\cos\theta+\pi\sin\theta)+2\pi i\zeta'(0)\cos^2\theta], \\ C^{+}_{12} &= -i\gamma e^{2i\theta}/4, \\ C^{-}_{11} &= (4\pi)^{-1}e^{i\theta}[\gamma(\log 4\sin\theta+\pi\cos\theta)+2\eta''(0)\cos^2\theta \\ &\qquad \times(2\pi e^{i\theta}+\pi\cos\theta+\log 4\sin\theta)-4\pi(i\pi+\log 2)\zeta'(0)\cos^2\theta, \\ C^{-}_{12} &= e^{i\theta}\cos^2\theta[\zeta'(0)-i\eta''(0)e^{i\theta}]/2. \end{aligned}\right\} \tag{6.16}$$

For case c (channel to wall flow) the $C^{\pm}_{0j}$ are still given by (6.15). From (5.7) and (5.8) we get the following values of $C^{\pm}_{1j}$:

$$\left.\begin{aligned} C^{+}_{11} &= \tfrac{1}{2}\gamma-2ie^{i\theta}\cos^2\theta[\zeta'(0)-\eta''(0)\sin\theta], \\ C^{+}_{12} &= \tfrac{1}{2}e^{i\theta}[\gamma\sin\theta+i\eta''(0)\cos^3\theta], \\ C^{-}_{11} &= (2\pi)^{-1}e^{i\theta}[\gamma(\pi\cos\theta+2\log 2\sin\theta)+(\zeta'(0)-\eta''(0)\sin\theta)\log 16\cos^2\theta], \\ C^{-}_{12} &= \tfrac{1}{2}\cos^2\theta[i\eta''(0)-\zeta'(0)e^{i\theta}]. \end{aligned}\right\} \tag{6.17}$$

By using the appropriate values of $C^{\pm}_{nj}$ in (6.11)–(6.14) we can obtain the initial conditions for the first two terms in the two outer expansions of the two flows joined by the inner expansion. The C^{+}_{nj} apply on one side of the junction, where $\phi > 0$, and the C^{-}_{nj} apply on the other side, where $\phi < 0$.

The $C^{\pm}_{nj}$ for case d will be given in the next section. When the outer expansion is a wall flow or channel flow, only (6.12) is needed, while when it is a jet (6.11)–(6.13) are needed. The unused conditions must then be satisfied automatically, so they serve as a check on the analysis.

7. Determination of z_0 for case d (jet splitting into two wall flows)

We shall now consider the analytic functions $z_k(\phi+i\psi)$ in the inner expansion (2.8) for case d of a jet hitting a wall and splitting into two wall flows. They are defined in the split strip shown in figure 3(d) and satisfy (3.1) or (3.2) on the free boundaries and (3.3) on the rigid boundary. However, since the dividing streamline $-\psi^0(\epsilon)$ depends upon ϵ, the K_k in (3.3) are modified for $k \geqslant 2$ by additional terms involving the derivatives of $\psi^0(\epsilon)$ at $\epsilon = 0$. Those derivatives can be determined by matching the resulting solution to the expansion of the incident flow. We shall not determine them because we shall consider explicitly only z_0 and z_1.

Now z_0 must satisfy $(x_0')^2+(y_0')^2 = 1$ on $\psi = 0$ and on $\psi = -1$ for all ϕ, and $y_0 = \eta'(0)x_0$ on $\psi = -\psi^0(0)$ for $\phi > 0$. Here $\psi^0(0)$ is unknown and must be found as part of the solution. This problem for z_0 and $\psi^0(0)$ is well known and has been solved explicitly, since it does not involve gravity and it involves a straight boundary. From Birkhoff & Zarantonello (1957), pages 35–36, we obtain

$$z_0 = \frac{e^{i\theta}}{\pi}\left\{i\pi(1+C)-2\delta S+C\log(\xi^2-2C\xi+1)-iS\log\left[\frac{\xi-e^{i\delta}}{\xi-e^{-i\delta}}\right]\right. \\ \left. +\log\left[\frac{\xi+1}{\xi-1}\right]-C\log(\xi^2-1)\right\}, \tag{7.1}$$

$$\psi^0(0) = \tfrac{1}{2}(1+\cos\delta). \tag{7.2}$$

Here $\xi = d(\phi + i\psi)/dz$ is the complex velocity, $C = \cos\delta$ and $S = \sin\delta$ where δ is the angle between the jet and the tangent to the wall. Thus (7.1) gives z_0 in terms of the complex velocity, so it is a first-order ordinary differential equation for $z_0(\phi + i\psi)$. In (7.1) the branch of the logarithm is defined by $\log z = \log|z| + i \arg z$, $-\pi \leqslant \arg z < \pi$.

For matching we need the asymptotic form of (7.1) for large $|\phi|$, which is given by (6.5) with $n = 0$. We find the $C^{\pm}_{0j}$, $j = 0, 1$ to be

$$C^{-}_{00} = \frac{e^{i\theta}}{\pi}\left\{i\pi(1+C) - 2S\delta + e^{i\delta}\log(2iS) + Ae^{-i\delta} + \log\left[\frac{e^{i\delta}+1}{e^{i\delta}-1}\right] - C\log(e^{2i\delta}-1)\right\},$$

where

$$A = -2\pi SC - \log(2iS) + (1+C)\left[\log(e^{i\delta}-1) + \frac{3i\pi}{2}\right] + (1-C)\log(e^{i\delta}+1), \tag{7.3}$$

$$C^{-}_{01} = e^{i(\theta-\delta)},$$

$$C^{+}_{00} = \frac{e^{i\theta}}{\pi}\left\{(1+C)(i\pi - A) + 2S(\pi-\delta) + C\log(1-C) + \log 2 - iS\log\left[\frac{1-e^{i\delta}}{1-e^{-i\delta}}\right]\right\}, \quad -\psi^0(0) < \psi \leqslant 0,$$

$$C^{+}_{01} = e^{i\theta}, \quad -\psi^0(0) < \psi \leqslant 0,$$

$$\tilde{C}^{+}_{00} = \frac{e^{i\theta}}{\pi}\left\{(1-C)A + 2S(\pi-\delta) + C\log(1+C) - \log 2 - iS\log\left[\frac{1+e^{i\delta}}{1+e^{-i\delta}}\right]\right\}, \quad -1 \leqslant \psi < -\psi^0(0),$$

$$\tilde{C}^{+}_{01} = -e^{i\theta}, \quad -1 \leqslant \psi < -\psi^0(0).$$

The functions z_k, $k \geqslant 1$, are determined in appendix B.

8. Summary

We shall now summarize our results for the four cases *a–d* shown in figure 2. For simplicity we shall choose the point $\alpha + i\beta$, which is at the junction in the z plane, to be the origin, so we set $\alpha = \beta = 0$. We shall also set $\phi_j(\epsilon) = 0$. For the case of a single junction these choices are convenient and involve no loss of generality. However, for flows with several junctions it is necessary to consider non-zero values of α, β and $\phi_j(\epsilon)$. The corresponding formulas can be obtained by modifying the present analysis slightly. They will be presented elsewhere, when they will be used to treat the flow shown in figure 4.

In each of the four cases there is an outer expansion for $\phi < 0$ and one inner expansion for $|\phi| \leqslant 1$. In each of cases *a–c* there is another outer expansion for $\phi > 0$, while in case *d* there are two such outer expansions. For cases *a–c* the first outer expansion is

$$z = x_0(\phi + i\epsilon\psi) + i\eta(x_0) + \epsilon[x_1(\phi + i\epsilon\psi)\{1 + i\eta'(x_0)\} + i\zeta(x_0)] + O(\epsilon^2), \quad \phi < 0. \tag{8.1}$$

In case *a* this represents a wall flow. For it $x_0(\phi)$ is the solution of (4.8) of part 1 with $x_0(0) = 0$, $y = \eta(x)$ is the equation of the wall, $x_1(\phi)$ is given by (4.15) of part 1 with $\phi_0 = 0$, $x_1(0) = -\sin\theta$, where $\theta = \tan^{-1}\eta'(0)$, and

$$\zeta(x_0) = \{1 + [\eta'(x_0)]^2\}^{\frac{1}{2}}\{1 - \gamma\eta(x_0)\}^{-\frac{1}{2}}. \tag{8.2}$$

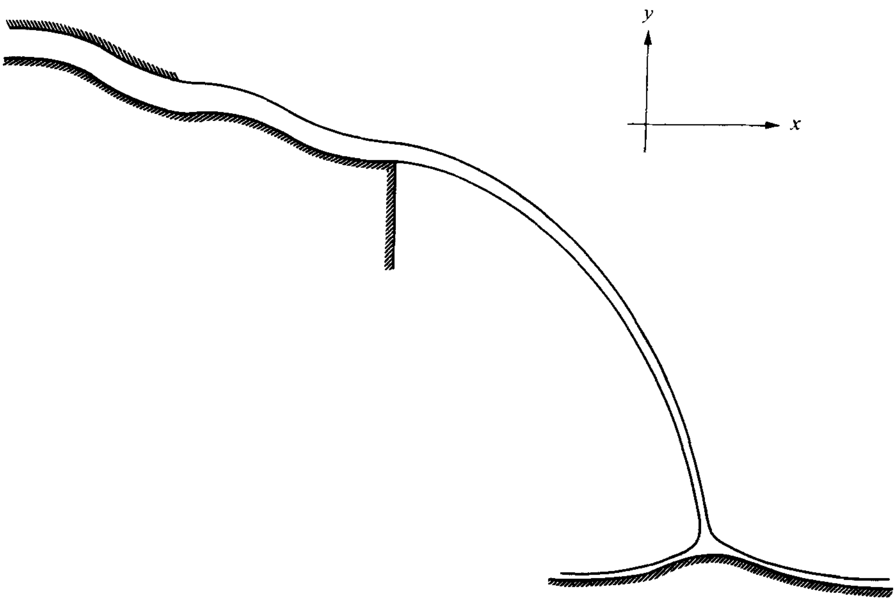

FIGURE 4. The free streamlines of a rather complicated flow with $\gamma = 1$ and $\epsilon = 0{\cdot}2$. This flow is described at the end of §8. The streamlines were determined by using the method of this paper, employing five outer and three inner expansions. Each was matched to the next one, starting with the channel flow at the upper left. These expansions describe the entire flow as well as the free streamlines.

In cases b and c (8.1) describes a channel flow. Here $x_0(\phi)$ is the solution of (5.11) of part 1 with $x_0(0) = 0$, $y = \eta(x)$ is the equation of the lower wall of the channel, $y = \eta(x) + \epsilon\zeta(x)$ is the equation of the upper wall and $x_1(\phi)$ is given by (5.15) of part 1 with $\phi_0 = 0$ and $x_1(0) = -\sin\theta$.

In all three cases the junction flow is given by the inner expansion

$$z = e^{i\theta}[\phi + i\epsilon\psi + i\epsilon] + \epsilon^2 z_1(\phi/\epsilon + i\psi) + \sum_{s=0}^{3} O(\epsilon^s\phi^{3-s}), \quad |\phi| \ll 1. \tag{8.3}$$

Here z_1 is given by (6.20) of part 1 in case a, by (4.9) in case b and by (5.6) in case c. We note that if $\phi = O(\epsilon)$ then the error in (8.3) is $O(\epsilon^3)$.

The second outer expansion in cases a and b describes a jet flow and is given by

$$z = x_0(\phi + i\epsilon\psi) + i(x_0\tan\theta - bx_0^2) + \epsilon[2bx_0 - \tan\theta + i]\,y_1(\phi + i\epsilon\psi) + O(\epsilon^2), \quad \phi > 0. \tag{8.4}$$

Here $x_0(\phi)$ is the real solution of the cubic equation (3.19) of part 1 with $a = 0$ and $b = \frac{1}{4}\gamma\sec^2\theta$ while y_1 is given by (3.32) of part 1 with these same values of a and b, and with $\beta = \theta$. The constants A_1 and B_1 in (3.32) are given by (3.34) and (3.35) in part 1 with $y_1(0) = \cos\theta$ and with the following value of $y_1'(0)$:

$$y_1'(0) = \frac{\gamma}{2}\sin 2\theta - \frac{\cos^2\theta\log 4}{2\pi}[2\eta''(0)\cos^2\theta + \gamma], \quad \text{case } a, \tag{8.5}$$

$$y_1'(0) = \frac{3\gamma}{8}\sin 2\theta - \frac{\gamma\cos^2\theta\log 4}{4\pi} + \frac{\cos^3\theta}{2\pi}[\pi\zeta'(0) - \eta''(0)(\cos\theta\log 4 + \pi\sin\theta)], \quad \text{case } b. \tag{8.6}$$

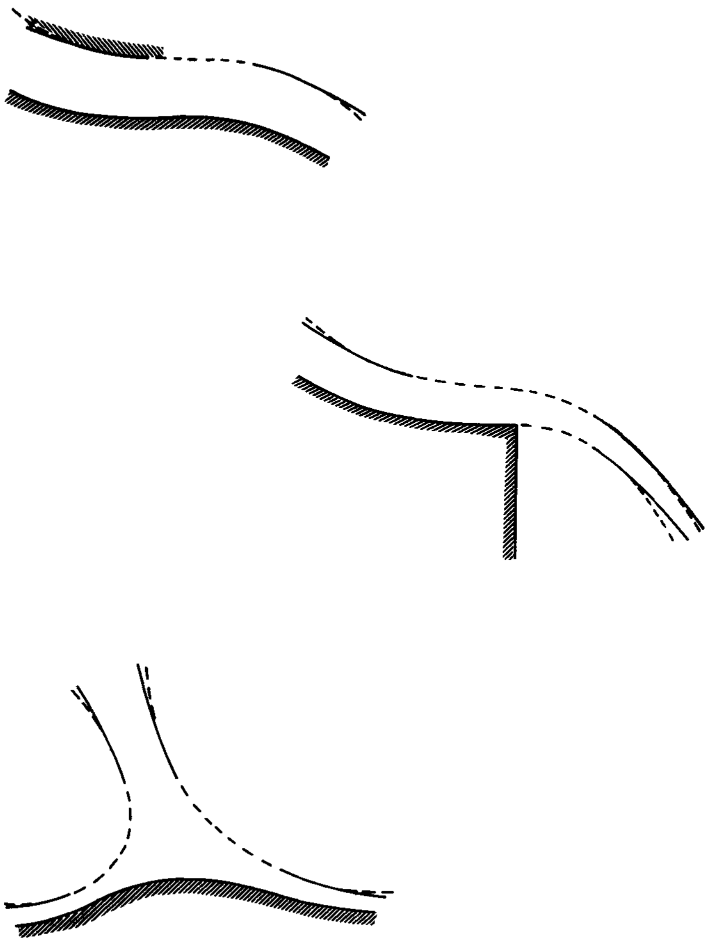

FIGURE 5. Enlarged drawings of the free streamlines near the three junctions in figure 4, showing the agreement between the results of the inner and outer expansions. The free streamlines given by each inner expansion are shown as dashed curves while those given by the outer expansions are shown as solid curves. The agreement between them is even better for smaller values of ϵ.

In case c the second outer expansion describes a wall flow which is given by (8.1) for $\phi > 0$ with all quantities defined as for case a just following that equation.

Now in case d the first outer expansion represents a jet flow. It is given for $\phi < 0$ by (8.4) with all quantities defined just after (8.4), but with the following changes: The angle θ is replaced by $\theta - \delta$, where δ is the angle between the jet and the wall,

$$y_1(0) = \operatorname{Im} C_{\overline{00}}^-$$

and $y_1'(0) = \operatorname{Im} C_{\overline{11}}^-$. The constants $C_{\overline{00}}^-$ and $C_{\overline{11}}^-$ are given by (7.3) and (B 6). The junction flow is given by an expansion of the form (8.3) with the first term replaced by $\epsilon z_0(\phi/\epsilon + i\psi)$ with z_0 given by (7.1) and z_1 by (4.8) in which p_1 and q_1 are defined by (B 3) and (B 4).

Beyond the junction the flow splits into two wall flows which are represented by two outer expansions of the form (8.1). One holds for $-1 \leqslant \psi \leqslant -\psi^0(\epsilon)$ and the other for $-\psi^0(\epsilon) \leqslant \psi \leqslant 0$, with $\psi^0(0) = \frac{1}{2}(1+\cos\delta)$. For the first flow $x_0(\phi)$ is the solution of (4.8) of part 1 using the negative value of the square root in that equation while for the second flow it is the solution for the positive value of this root. In both cases $x_0(0) = 0$ and $y = \eta(x)$ is the equation of the wall. For the first flow the ϵ term in (8.1) must be multiplied by $1-\psi^0$ while for the second flow it must be multiplied by ψ^0. In both cases $x_1(\phi)$ is given by (4.15) of part 1 and $\zeta(x_0)$ by (8.2). The initial values $x_1(0)$ in the two cases are

$$x_1(0) = (1-\psi^0)^{-1}\operatorname{Re}\tilde{C}^+_{00}, \quad -1 \leqslant \psi \leqslant -\psi^0 \tag{8.7}$$

and

$$x_1(0) = (\psi^0)^{-1}\operatorname{Re}C^+_{00}, \quad -\psi^0 \leqslant \psi \leqslant 0.$$

The constants C^+_{00} and $\tilde{C}^+_{00}$ are given in (7.3).

To illustrate the utility of the present method, we have applied it to the complicated flow shown in figure 4. The stream flows through a curved channel, emerges from the channel and continues along a curved stream bed, flows off the end of the bed to become a falling jet, hits a curved surface and splits into two streams which flow along the surface in opposite directions. To describe this flow we use a channel flow, a channel to wall junction flow, a wall flow, a wall to jet junction flow, a jet flow, a jet to two wall flows junction flow and two wall flows. We specify the upstream conditions in the channel flow and determine the other flows successively by matching each one to the next. In doing so we use all three outer expansions and three of the four inner expansions, utilizing all but case b, the channel to jet junction flow. The details of this calculation, together with several other examples, will be presented elsewhere.

In figure 4 the free streamlines are shown. In figure 5 enlargements of the three junctions are shown to indicate the agreement between the free streamlines determined from adjacent flows. The constants were taken to be $\gamma = 1$ and $\epsilon = 0{\cdot}2$. Thus the Froude number based on the channel width is $\epsilon^{-1} = 5$.

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Appendix A. Eigensolutions

For $k \geqslant 1$ the functions $w'_k = e^{-i\theta}z'_k$, which we have determined for the various cases, are not defined uniquely by the conditions which we have imposed upon them. Therefore, as we shall now show, the homogeneous form of the problem defining each w'_k possesses non-trivial solutions. We shall present these eigensolutions here, and examine their behaviour. We shall show that each eigensolution is singular either at a finite point or at $\phi = \pm\infty$.

This singular behaviour indicates what conditions we should impose upon w'_k in order to obtain a unique solution. By requiring w'_k to be bounded at finite points, we exclude the eigenfunctions which become infinite at such points. By requiring w'_k to grow no faster than a power of ϕ as ϕ tends to $\pm\infty$, we exclude the eigenfunctions which are singular at either $\pm\infty$, since they grow exponentially. In this way we exclude all the eigenfunctions and obtain a unique solution. That these conditions are the proper ones to impose is verified by the fact that the resulting inner expansion matches the two adjacent outer expansions.

Case a (wall flow to jet). In the plane of $s = s_1 + is_2$ an eigensolution for $w'_k = e^{i\theta}z'_k$ in case a is any function $g(s)$ analytic in the first quadrant and satisfying there the boundary conditions

$$\operatorname{Re} g = 0 \quad \text{on} \quad s_2 = 0, \quad \operatorname{Im} g = 0 \quad \text{on} \quad s_1 = 0. \tag{A 1}$$

It is easy to verify that the eigenfunctions are

$$g_n(s) = is^{2n+1}, \quad n = 0, \pm 1, \ldots, \tag{A 2}$$

$$h_n(s) = i[(s-1)^{-2n-1} + (s+1)^{-2n-1}], \quad n = 0, 1, 2, \ldots . \tag{A 3}$$

The corresponding eigenfunctions for w'_k in the strip $-1 < \psi < 0$ of the plane of $f = \phi + i\psi$ are obtained from (A 2) and (A 3) by setting $s = (1 + e^{-\pi f})^{\frac{1}{2}}$. Upon integrating $e^{i\theta} w'_k(f)$ we obtain the following eigensolutions for z_k:

$$G_n(\phi + i\psi) = i\, e^{i\theta} \int_{-i}^{\phi + i\psi} (1 + e^{-\pi f})^{n+\frac{1}{2}} df, \quad n = 0, \pm 1, \ldots, \tag{A 4}$$

$$H_n(\phi + i\psi) = i\, e^{i\theta} \int_{-i}^{\phi + i\psi} \{[(1 + e^{-\pi f})^{\frac{1}{2}} - 1]^{-2n-1} + [(1 + e^{-\pi f})^{\frac{1}{2}} + 1]^{-2n+1}\} df, \quad n = 0, 1, \ldots . \tag{A 5}$$

From (A 2) we see that, for $n \geqslant 0$, $g_n(s)$ becomes infinite as $s \to \infty$, corresponding to upstream infinity in the flow. Similarly, (A 4) shows that G_n grows like $e^{-(n+\frac{1}{2})\pi\phi}$ as ϕ tends to $-\infty$. We also see that, for $n < 0$, $g_n(s)$ becomes infinite at $s = 0$, corresponding to the point where the flow leaves the wall. The corresponding G_n given by (A 4) becomes infinite at $\phi = 0$, $\psi = -1$. The function h_n given by (A 3) is singular at $s = 1$, which corresponds to a point infinitely far downstream, and H_n grows exponentially as $\phi \to +\infty$. This singular behaviour shows that in this case all the eigenfunctions are excluded by the boundedness and growth conditions presented above.

Case b (channel to jet). The eigensolutions in case b in the s plane are

$$g_n(s) = is^{2n+1}, \quad n = 0, \pm 1, \ldots, \tag{A 6}$$

$$h_n(s) = i[(s-1)^{-2n-1} + (s+1)^{-2n-1}], \quad n = 0, 1, \ldots, \tag{A 7}$$

$$k_n(s) = i[(s-i)^{-2n-1} + (s+i)^{-2n-1}], \quad n = 0, 1, \ldots . \tag{A 8}$$

For $n \geqslant 0$, $g_n(s)$ is singular at $s = \infty$, which corresponds to the upper edge of the channel. For $n < 0$, g_n is singular at $s = 0$, which corresponds to the lower edge of the channel. For $n \geqslant 0$, h_n is singular at $s = 1$, which corresponds to downstream infinity. For $n \geqslant 0$, k_n is singular at $s = i$, which corresponds to upstream infinity.

The corresponding eigenfunctions for $e^{-i\theta} z_k$ are obtained by expressing s in terms of f by (4.5) in (A 6)–(A 8) and then integrating with respect to f from $-i$ to $\phi + i\psi$. The resulting eigenfunctions are singular at the points corresponding to the singularities of the integrands, in such a way that they are excluded by the preceding conditions.

Case c (channel to wall flow). The eigenfunctions for w'_k in case c, are, in the f plane,

$$g_n(f) = i(e^{\pi f} - 1)^{n+\frac{1}{2}}, \quad n = 0, \pm 1, \ldots, \tag{A 9}$$

$$h_n(f) = i\{[(e^{\pi f} - 1)^{\frac{1}{2}} - i]^{-2n-1} + [(e^{\pi f} - 1)^{\frac{1}{2}} + i]^{-2n-1}\}, \quad n = 0, 1, \ldots . \tag{A 10}$$

The g_n with $n \geqslant 0$ are singular as $\mathrm{Re} f \to \infty$, i.e. far downstream. Those g_n with $n < 0$ are singular at the upper edge of the channel, $f = 0$. The h_n, $n \geqslant 0$, are singular as $\mathrm{Re} f \to -\infty$, i.e. far upstream. The corresponding eigenfunctions for $e^{-i\theta} z_k$, obtained by integrating g_n and h_n as in case b, are also singular where the integrands are singular, and are also excluded by the preceding conditions.

Case d (jet hitting a wall). The eigenfunctions for case d, in the s plane, are given by (A 6), (A 7) and (A 8) with $s \pm i$ replaced by $s \pm \tilde{a} i$ where $\tilde{a} = [\psi^0/(1-\psi^0)]^{\frac{1}{2}}$. For $n \geqslant 0$, g_n is singular at $s = \infty$, i.e. at downstream infinity where $-1 \leqslant \psi \leqslant -\psi^0$. For $n < 0$, g_n is singular at $s = 0$, i.e. at downstream infinity in the interval

$$-\psi^0 \leqslant \psi \leqslant 0.$$

For $n \geqslant 0$, h_n is singular at $s = 1$, which corresponds to upstream infinity. For $n \geqslant 0$, k_n is singular at $s = \tilde{a} i$, which is the stagnation point.

The eigenfunction $G_n(\phi + i\psi)$ of z_k is obtained from g_n given by (A 6) *via* the integral

$$G_n(\phi + i\psi) = i\, e^{i\theta} \int_{-i}^{\phi + i\psi} s^{2n+1} \zeta^{-1} df, \quad n = 0, \pm 1, \ldots . \tag{A 11}$$

Here $\zeta = d(\phi + i\psi)/dz$, where z is given by (7.1), and s is determined as a function of f by (B 1) and (B 2), in which u is a parameter. The eigenfunctions corresponding to (A 7) and (A 8) are obtained by replacing g_n by h_n or k_n in (A 11). All these eigenfunctions have singularities which are not allowed by the preceding conditions.

Appendix B. Calculation of z_k, $k \geqslant 1$, for case d

The analytic function z_k, $k \geqslant 1$, for case d satisfies (3.2) on the free boundaries $\psi = 0$ and $\psi = -1$, and (3.3) on the rigid boundary $\psi = -\psi^0(0)$, $\phi > 0$. To find it we introduce the new analytic function $w_k = e^{-i\theta} \xi z_k'$. For it the boundary conditions become $\mathrm{Re}\, w_k = J_k$ on $\psi = 0$ and $\psi = -1$, and $\mathrm{Im}\, w_k = (x_0')^{-1} \cos^2\theta K_k'$ on the two sides of the slit $\psi = -\psi^0(0)$, $\phi > 0$. To solve for w_k we map the slit strip $-1 \leqslant \psi \leqslant 0$ on the first quadrant of the s-plane by the two mappings

$$\phi + i\psi = -\frac{1}{\pi}\left\{\psi^0 \log\left(\frac{u-1}{2\psi^0}\right) + (1-\psi^0)\log\left(\frac{u+1}{2[1-\psi^0]}\right)\right\}, \tag{B 1}$$

$$s = [(u-1)/(u+1)]^{\frac{1}{2}}. \tag{B 2}$$

Here and below ψ^0 denotes $\psi^0(0)$.

As a consequence of this mapping, $\mathrm{Re}\, w_k$ is specified along the positive real axis and $\mathrm{Im}\, w_k$ is specified along the positive imaginary axis of the s plane. This problem can be solved with the result that w_k is given by the right side of (4.6). Then z_k is given by (4.8) in which s is defined in terms of $f = \phi + i\psi$ by (B 1) and (B 2), and with the factor $\xi^{-1}(f)$ inserted before the second integral. The functions $p_k(\sigma)$ and $q_k(\sigma)$ in the integrand are defined as follows with $\psi^0 = (\psi^0)^-$:

$$p_k(\sigma) = (x_0')^{-1}\cos^2\theta K_k'\left(-\frac{1}{\pi}\left[\psi^0 \log\left(\frac{\sigma^2}{(1+\sigma^2)\,\psi^0}\right) - (1-\psi^0)\log(1-\sigma^2)(1-\psi^0)\right], \psi^0\right),$$
$$\left(\frac{1-\psi^0}{\psi^0}\right)^{\frac{1}{2}} < \sigma. \tag{B 3}$$

For $0 < \sigma < [(1-\psi^0)/\psi^0]^{\frac{1}{2}}$, $p_k(\sigma)$ is given by (B 3) with $(\psi^0)^-$ replaced by $(\psi^0)^+$.

$$q_k(\sigma) = J_k\left(-\frac{1}{\pi}\left[\psi^0 \log\frac{\sigma^2}{(1-\sigma^2)\psi^0} - (1-\psi^0)\log(1-\sigma^2)(1-\psi^0)\right], 0\right), \quad 0 < \sigma < 1. \tag{B 4}$$

For $\sigma > 1$, $q_k(\sigma)$ is given by (B 4) with $\psi = 0$ replaced by $\psi = -1$.

For $k = 1$, J_1 and K_1 are given below (4.8). Then the coefficients in the asymptotic form (6.5) of z_1 can be found by using these values and the above formulas in (4.8). In particular we find

$$C_{\overline{12}}^- = -\tfrac{1}{4}i\gamma e^{2i(\theta-\delta)}, \tag{B 5}$$

$$C_{\overline{11}}^- = \frac{2i\,e^{i(\theta-\delta)}}{\pi}\int_0^\infty \left\{\frac{p_1(\sigma)}{1+\sigma^2} + \frac{\tilde{q}_1(\sigma)}{1-\sigma^2}\right\} ds + \frac{\gamma e^{i(\theta-\delta)}}{2}\left\{\frac{i\,e^{i(\theta-\delta)}}{\pi}\log 2 + \operatorname{Im} C_{\overline{00}}^- \right.$$
$$\left. + \sin(\theta-\delta)\left[\log 2 - \psi^0(i+\log(2\psi^0)) + (\psi^0-1)\log 2(1-\psi^0)\right]\right\}. \tag{B 6}$$

In (B 6) $C_{\overline{00}}^-$ is given by (7.3) and $\tilde{q}_1(\sigma)$ is defined by

$$\tilde{q}_1(\sigma) = q_1(\sigma) - \frac{\gamma}{2}\left\{\frac{1}{\pi}\sin(\theta-\delta)\left[\log|1-\sigma^2| - \psi^0\log\sigma^2 - \log 2\right]\right.$$
$$\left. - \cos(\theta-\delta)H(\sigma-1) + \operatorname{Im} C_{\overline{00}}^-\right\}. \tag{B 7}$$

Here $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. We note that $\tilde{q}_1(\sigma) = O(|1-\sigma|)$ for σ near 1, so the integral in (B 7) is finite.

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